BOUNDARY ELEMENT METHOD
FOR INTERNAL AXISYMMETRIC FLOW

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Superposition of flows in a hydroturbine

1. Axisym., irrot.
2. Rot.
3. Blades + Comp.
Formulation of the Problem

Hypotheses for the flow $v$:  
(i) The flow is incompressible, i.e. $\nabla \cdot \mathbf{v} = 0$;  
(ii) The flow is potential, i.e. there exists a scalar potential $\varphi$ with $\mathbf{v} = \nabla \varphi$;  
(iii) The flow is axisymmetric, i.e. $\mathbf{v} = \mathbf{v}(r, z)$;  
(iv) The flow is irrotational, i.e. the circumferential component of velocity $v_\theta = 0$.

Hypotheses for the geometry of the passage:  
(v) The passage is a domain of revolution in $\mathbb{R}^3$ generated by a simply connected domain $\Omega \subseteq \{(r, z) \in \mathbb{R}^2 \colon r \geq 0\}$;  
(vi) The boundary of the passage is the union of two disjoint $C^1$ surfaces of revolution.  
The inner surface represents the crown and the outer surface represents the bend.  
(vii) The inlet and the exit belong to the following types:

![Diagram of the passage types](image)

Governing equation $\nabla^2 \varphi = 0$:  
\[
\frac{\partial^2 \varphi}{\partial z^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial r^2} = 0, \quad v_r = \frac{\partial \varphi}{\partial r}, \quad v_z = \frac{\partial \varphi}{\partial z}, \quad v_\theta = 0
\]

Boundary conditions:  
(i) At the wall we have the normal component of velocity $v_n = \mathbf{n} \cdot \mathbf{v} = 0$.  
(ii) At infinity we impose the following asymptotic conditions:  
(a) Radial with distance between the planes $b_0$: $v_z \ll v_r \sim \pm \frac{q}{2 \pi r b_0}$ as $r \to \infty$ between the planes forming the passage;  
(b) Cylinder with radius $r_0$: $v_r \to 0$, $v_z \to \pm \frac{q}{2 \pi r_0}$ as $z \to \pm \infty$ inside the cylinder;  
(c) Annular cylinder with radii $r_0 < r_1 < r_0$: $v_r \to 0$, $v_z \to \pm \frac{q}{\pi \left(r_0^2 - r_1^2\right)}$ as $z \to \pm \infty$ inside the annulus;  
(d) Cone with vertex $(0, z_v)$ and generatrix $r = (z - z_v) \tan \alpha$: $v_\rho \sim \pm \frac{q}{\pi \sin^2 \alpha \rho^2}$,  
as $\rho \to \infty$, where $\rho = |(r, z - z_v)|$ inside the cone.
Biot-Savart

\[ V = \frac{1}{4\pi} \frac{\gamma \times \rho}{|\rho|^3} \]

\[ R = |\rho|^2 \]
Theory

Theorem A. The induced velocity at a test point $\mathbf{r}_c = (0, r_c, z_c)$ due to axisymmetric distributed surface vorticity $\gamma(\ell)$ on the bounding surfaces of revolution is

$$v_r = \frac{1}{4\pi} \int_{\mathcal{L}} \gamma(\ell) r(\ell) (z_c - z(\ell)) \int_0^{2\pi} \frac{\sin \theta}{R(r(\ell), z(\ell), \theta)^3} d\theta d\ell,$$

$$v_z = \frac{1}{4\pi} \int_{\mathcal{L}} \gamma(\ell) r(\ell) \int_0^{2\pi} \frac{r(\ell) - r_c \sin \theta}{R(r(\ell), z(\ell), \theta)^3} d\theta d\ell,$$

$$v_\theta = 0,$$

where

$$R(r, z, \theta) = r^2 + r_c^2 - 2r \cdot r_c \sin \theta + (z - z_c)^2,$$

and $\mathcal{L}$ denotes the union of curves that generate the boundary and $\ell$ is arclength along these curves.

Theorem B. The boundary condition $v_n = 0$ is equivalent to $v_r = \gamma$, where $v_r$ is the tangential component of velocity inside the wall. (see next page)

Theorem C. Axisymmetric distributed surface vorticity $\gamma(\ell)$ on the bounding surfaces of revolution satisfies a Fredholm integral equation of the second kind

$$\frac{\gamma(\ell_c)}{2} = \frac{1}{4\pi} \int_{\mathcal{L}} \gamma(\ell) r(\ell) \int_0^{2\pi} \frac{r'(\ell_c)(z(\ell_c) - z(\ell)) \sin \theta + \frac{z'(\ell_c)}{r(\ell_c)} (r(\ell) - r_c \sin \theta)}{(r(\ell)^2 + r_c^2)^2 - 2r(\ell_c)r(\ell) \sin \theta + (z(\ell_c) - z(\ell))^2 R_{\ell}} d\theta d\ell,$$

where prime denotes differentiation with respect to arclength $\ell$.

Theorem D. Let $q$ be constant. If $\gamma$ satisfies the following a priori conditions for the different cases of inlet/exit:

(a) $\gamma = \pm \frac{q}{2\pi b_0 r}$ on two horizontal planes, a distance $b_0$ apart with opposite signs on the two planes,

(b) $\gamma = \frac{q}{\pi r_0^2}$ on a vertical cylinder with radius $r_0$,

(c) $\gamma = \pm \frac{q}{\pi (r_0^2 - r_1^2)}$ on a vertical annular cylinder with radii $0 < r_1 < r_0$, with opposite signs at $r_1$ and $r_0$,

(d) $\gamma = \pm \frac{q}{\pi \sin^2 \alpha \rho^2}$, where $\rho = ||(r, z - z_v)||$, on a cone with generatrix $r = (z - z_v) \tan \alpha$,

then the induced velocity field has flux that is asymptotic to $q$ and asymptotically satisfies the boundary conditions at infinity (inlet/exit).

Theorem E. At the intersection of the inner bounding surface with the $z$ axis $\gamma = 0$. 
Proof of Theorem B

\[ \text{Stokes } \Rightarrow \quad v_{\tau} h - v_{\tau}^* h = \gamma h = \gamma h \]

\[ \lim_{h \to 0} \quad v_{\tau} - v_{\tau}^* = \gamma \]
Computation

Geometry + flow rate
\[ \Downarrow \text{Representation & integration} \]

Coefficients
\[ \Downarrow \text{Linear solve (direct)} \]

\( \gamma \) on the boundary

Test points: Located at the vertices of the boundary elements.

Representation and integration:

<table>
<thead>
<tr>
<th>Type of boundary element</th>
<th>Geometrical representation of the generatrix</th>
<th>Representation of vortex density ( \gamma )</th>
<th>Integration</th>
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<tbody>
<tr>
<td>General</td>
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<td>linear spline</td>
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<td>( \theta ): numerical</td>
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<td>Semi-infinite</td>
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<td>( r ): “closed form”</td>
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<td></td>
<td></td>
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<td>( \theta ): numerical</td>
</tr>
</tbody>
</table>

Linear system: We solve a linear system of equations for the unknown nodal values \( \gamma_i \).
Boundary elements not incident to the test point

**Geometry:**  \( r(\ell) = a_r + b_r \ell, \ z(\ell) = a_z + b_z \ell \)

where \( a_r = r_i - m_r \ell_i, \ b_r = m_r \ell_i, \ a_z = z_i - m_z \ell_i, \ b_z = m_z \ell_i, \)
\( r(\ell_i) = r_i, \ z(\ell_i) = z_i \) is the starting point and \( m_r^2 + m_z^2 = 1. \)

**Vorticity:**  \( \gamma(\ell) = A + B \ell, \)

where \( A = \gamma_i - \frac{\ell_i (\gamma_{i+1} - \gamma_i)}{\ell_{i+1} - \ell_i}, \ B = \frac{\gamma_{i+1} - \gamma_i}{\ell_{i+1} - \ell_i}, \)
\( t = (\ell - \ell_i)/(\ell_{i+1} - \ell_i) \) and \( \gamma_i \) are the nodal values of \( \gamma. \)

**Integration:**

\[
\begin{align*}
  u_r &= \frac{1}{4\pi} \int_0^{2\pi} \int_{\ell_i}^{\ell_{i+1}} \frac{\bar{a}_r + \bar{b}_r \ell + \bar{c}_r \ell^2 + \bar{d}_r \ell^3}{(a + b \ell + c \ell^2)^{\frac{3}{2}}} \, d\ell \, d\theta = \frac{1}{4\pi} \int_0^{2\pi} \left( \bar{a}_r T_0^* + \bar{b}_r T_1^* + \bar{c}_r T_2^* + \bar{d}_r T_3^* \right) \, d\theta \\
  u_z &= \frac{1}{4\pi} \int_0^{2\pi} \int_{\ell_i}^{\ell_{i+1}} \frac{\bar{a}_z + \bar{b}_z \ell + \bar{c}_z \ell^2 + \bar{d}_z \ell^3}{(a + b \ell + c \ell^2)^{\frac{3}{2}}} \, d\ell \, d\theta = \frac{1}{4\pi} \int_0^{2\pi} \left( \bar{a}_z T_0^* + \bar{b}_z T_1^* + \bar{c}_z T_2^* + \bar{d}_z T_3^* \right) \, d\theta
\end{align*}
\]

where

\[
\begin{align*}
  T_0^* &= T_n(\ell_{i+1}) - T_n(\ell_i) \\
  T_0 &= \frac{2}{(4a - b^2)(\ell^2 + b\ell + a)^{\frac{3}{2}}} \\
  T_1 &= -\frac{2}{(4a - b^2)(\ell^2 + b\ell + a)^{\frac{3}{2}}} \\
  T_2 &= -\frac{(3a - b^2)\ell - ab}{(4a - b^2)(4a - b^2)(\ell^2 + b\ell + a)^{\frac{3}{2}}} + \ln \left( \frac{2}{3b} \left( (\ell^2 + b\ell + a)^{\frac{3}{2}} + 2\ell + b \right) \right) \\
  T_3 &= \frac{(4a - b^2)\ell^2 + b(10a - 3b^2)\ell + a(8a - 3b^2)}{(4a - b^2)(\ell^2 + b\ell + a)^{\frac{3}{2}}} - \frac{3b}{2} \ln \left( \frac{2}{3b} \left( (\ell^2 + b\ell + a)^{\frac{3}{2}} + 2\ell + b \right) \right)
\end{align*}
\]

\[
\begin{align*}
  a &= a_r^2 + r_r^2 - 2r_r a_r \sin \theta + (z_r - a_z)^2 \\
  b &= 2(a_r b_r - r_r b_r \sin \theta - (z_r - a_z)b_z) \\
  \bar{a}_r &= A a_r (z_r - a_z) \sin \theta \\
  \bar{b}_r &= (B a_r) (z_r - a_z) + (A b_r) (z_r - a_z) - (A a_r) b_z \sin \theta \\
  \bar{c}_r &= (B b_r) (z_r - a_z) - (B a_r) b_z - (A b_r) \sin \theta \\
  \bar{d}_r &= -B b_r b_z \sin \theta \\
  \bar{a}_z &= A a_r (a_r - r_r \sin \theta) \\
  \bar{b}_z &= (B a_r) (a_r - r_r \sin \theta) + (A b_r) (a_r - r_r \sin \theta) + (A a_r) b_r \\
  \bar{c}_z &= (B b_r) (a_r - r_r \sin \theta) + (B a_r) b_r + (A b_r)^2 \\
  \bar{d}_z &= B b_r^2
\end{align*}
\]

Note that the coefficient of \( \ell^2 \) in the denominator of the integrands is \( b_r^2 + b_z^2 = 1. \)
Integrals over elements incident to the test point

For $\ell_{j-1} \leq \ell \leq \ell_j$, $\Delta \ell_{j-1} = \ell_j - \ell_{j-1}$, and $j = i, i + 1$

$$r(t) = r_{j-1} H_{01}(t) + r_j H_{02}(t) + \Delta \ell_{j-1} \left( r'_{j-1} H_{11}(t) + r'_j H_{12}(t) \right),$$

$$z(t) = z_{j-1} H_{01}(t) + z_j H_{02}(t) + \Delta \ell_{j-1} \left( z'_{j-1} H_{11}(t) + z'_j H_{12}(t) \right),$$

$$\gamma(t) = \gamma_{j-1}(1 - t) + \gamma_j t,$$

$$H_{01}(x) = 1 - 3 x^2 + 2 x^3, \quad H_{02}(x) = 3 x^2 - 2 x^3,$$

$$H_{11}(x) = x - 2 x^2 + x^3, \quad H_{12}(x) = -x^2 + x^3,$$

Integral: $s = \ell - \ell_i$

$$u_r = \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\Delta \ell_{i-1}}^{\Delta \ell_i} \frac{r'o_{0z} \sin \theta + z'o_{0z} r (r - r_o \sin \theta)}{(r^2 + r_o^2 - 2 r r_o \sin \theta + (z_o - z)^2)^{\frac{3}{2}}} \, ds \, d\theta$$

Taylor expansions: $\delta = \frac{\pi}{2} - \theta$

$$\sin \theta = 1 - \frac{1}{2} \delta^2 + O(\delta^4), \quad r = r_0 + r'_0 s + \frac{1}{2} r''_0 s^2 + \frac{1}{6} r'''_0 s^3,$$

$$\gamma = \gamma_0 + \gamma'_0 s,$$

$$z = z_0 + z'_0 s + \frac{1}{2} z''_0 s^2 + \frac{1}{6} z'''_0 s^3.$$

**Theorem F.** Let $I$ denote the integrand. Then with the above notation

$$I = \frac{r'_0 \gamma r(z_o - z) \sin \theta + z'_0 \gamma r (r - r_o \sin \theta)}{(r^2 + r_o^2 - 2 r r_o \sin \theta + (z_o - z)^2)^{\frac{3}{2}}} = \frac{\gamma_0 r_0 A(\varphi)}{\rho} + T(\varphi) + O(\rho),$$

where $\rho \cos \varphi = r_0 \delta$, $\rho \sin \varphi = s$, and

$$A(\varphi) = \frac{1}{2} \left( z'_0 r'_0 - r'_0 z'_0 \right) \sin^2 \varphi + \frac{z'_0}{2 r_0} \cos^2 \varphi,$$

$$T(\varphi) = \sin \varphi (\gamma'_0 + r'_0 \gamma_0) A(\varphi) + \gamma_0 r_0 \left[ \frac{r'_0}{2 r_0} \cos^2 \varphi \sin \varphi + \frac{1}{6} (z'_0 r''_0 - r'_0 z''_0) \sin^3 \varphi \right.$$

$$\left. - \frac{3}{2} \frac{r'_0}{r_0} \cos^2 \varphi \sin \varphi + \frac{1}{2} (r''_0 r'_0 + z'''_0) \sin^3 \varphi \right) A(\varphi).$$
Decomposition of the integrand into regular and principal parts:

\[ I = \frac{\gamma_0 r_0 A(\varphi)}{\rho} + \left( I - \frac{\gamma_0 r_0 A(\varphi)}{\rho} \right) \]

\[ \lim_{\rho \to 0} \left( I - \frac{\gamma_0 r_0 A(\varphi)}{\rho} \right) = T(\varphi) \Rightarrow \text{regular part} \to \text{proper integral} \Rightarrow \text{numerical evaluation} \]

Theorem G. Let \( P \) denote the contribution to \( u_+ \) of the principal part of \( I \).

(a) If \( \ell_i \) is an interior point of a cubic spline representing the boundary generatrix, then

\[ P = \frac{\gamma_0 r_0}{2\pi} \left( -\Delta \ell_{i-1} J_1 \left|_{-\frac{\varphi_+}{2}}^{\varphi_-} \right. + \pi r_0 J_2 \left|_{-\varphi_+}^{\varphi_+} \right. + \Delta \ell_i J_1 \left|_{\varphi_-}^{\varphi_+} \right. \right), \]

where \( \varphi_- = \text{atan}2(\pi r_0, \Delta \ell_{i-1}) \), \( \varphi_+ = \text{atan}2(\pi r_0, \Delta \ell_i) \), and

\[ J_1(\varphi) = -\frac{1}{2} (z_0^r r_0'' - r_0^l z_0'') \cos \varphi + \frac{z_0'}{2 r_0} \left( \frac{1}{2} \log \left( \frac{1 - \cos \varphi}{1 + \cos \varphi} \right) + \cos \varphi \right), \]

\[ J_2(\varphi) = \frac{1}{2} (z_0^r r_0'' - r_0^l z_0'') \left( \frac{1}{2} \log \left( \frac{1 + \sin \varphi}{1 - \sin \varphi} \right) - \sin \varphi \right) + \frac{z_0'}{2 r_0} \sin \varphi. \]

(b) If \( \ell_i \) is the juncture of arcs with different radii or an arc and a straight line segment, then

\[ P = \frac{\gamma_0 r_0}{2\pi} \left( -\Delta \ell_{i-1} J_1 \left|_{-\frac{\varphi_+}{2}}^{\varphi_-} \right. + \pi r_0 J_2 \left|_{0}^{\varphi_-} \right. + \pi r_0 J_2 \left|_{0}^{\varphi_+} \right. + \Delta \ell_i J_1 \left|_{\varphi_-}^{\varphi_+} \right. \right), \]

where \( J_{2\pm} \) are constructed the same way as \( J_2 \), but using left or right second derivatives \( r_0^{\prime\prime} \) and \( z_0^{\prime\prime} \).
Improper integrals at the inlet and exit

Test point: \((r_c, z_c)\).

**Radial:** (level \(z_0\))

\[
v_r = \frac{\gamma_0 h_c}{4\pi} \int_0^{2\pi} \frac{\sin \theta}{h_c^2 + r_c^2 \cos^2 \theta} \left(1 - \frac{\ell_0 - r_c \sin \theta}{R(\ell_0, z_0, \theta)^2}\right) d\theta,
\]

\[
v_z = \frac{\gamma_0}{4\pi} \int_0^{2\pi} \frac{d\theta}{R(\ell_0, z_0, \theta)^2},
\]

where \(h_c = z_0 - z_0\).

**Cylindrical:** (radius \(r_0\))

\[
v_r = -\frac{\gamma_0 r_0}{4\pi} \int_0^{2\pi} \frac{\sin \theta \, d\theta}{R(r_0, \pm \ell_0, \theta)^2},
\]

\[
v_z = \frac{\gamma_0 r_0}{4\pi} \int_0^{2\pi} \frac{r_0 - r_c \sin \theta}{r_c^2 - 2r_c r_0 \sin \theta + r_0^2} \left(\pm 1 - \frac{\pm \ell_0 - z_c}{R(r_0, \pm \ell_0, \theta)^2}\right) d\theta,
\]

**Conical:** (vertex \((0, z_v)\), angle \(\alpha\))

\[
v_r = \frac{\gamma_0 \sin \alpha}{4\pi s_c^2} \int_0^{2\pi} \frac{\sin \theta}{1 - \sigma^2} \left(\frac{\tau_r - \frac{\tau_0 + s_c \mu_r}{R_0^\beta}}{R_0^\beta}\right) \sin \theta \, d\theta,
\]

\[
v_z = \frac{\gamma_0 \sin \alpha}{4\pi s_c^2} \int_0^{2\pi} \frac{1}{1 - \sigma^2} \left(\frac{\tau_z - \frac{\tau_0 + s_c \mu_z}{R_0^\beta}}{R_0^\beta}\right) - \lambda \sin \beta \sin \theta \right\} d\theta,
\]

where the distance and angle from the cone vertex to the test point are \((s_c, \beta)\),

\[
\sigma = \sin \alpha \sin \beta \sin \theta + \cos \alpha \cos \beta,
\]

\[
R_0 = R(\ell_0 \sin \alpha, z_0 + \ell_0 \cos \alpha, \theta) = s_c^2 - 2s_c \ell_0 \sigma + \ell_0^2,
\]

\[
\lambda = \log \frac{s_c - \sigma \ell_0 + R_0^\beta}{(1 - \sigma)\ell_0^\beta},
\]

\[
\tau_r = \sigma \cos \beta - \cos \alpha,
\]

\[
\tau_z = \sin \alpha - \sigma \sin \beta \sin \theta,
\]

\[
\mu_r = \sigma \cos \alpha - \cos \beta (2\sigma^2 - 1),
\]

\[
\mu_z = \sigma \sin \alpha - \sin \beta \sin \theta (2\sigma^2 - 1),
\]
Example: Butt Valley (Pacific Gas & Electric)

Streamlines and equipotentials, 35 nodes per surface (70 × 70 linear system)

**Internal verification:** Relative errors in flow rate and potential.

**Speed:** Not optimized FORTRAN 77 code.

<table>
<thead>
<tr>
<th>CPU</th>
<th>OS</th>
<th>Compiler</th>
<th>Time</th>
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