3.1 \(X\) is contractible \(\iff f \simeq g\) for all \(Y\) and \(f, g: Y \to X\).

\(\iff\): Let \(Y = X\), \(f(x) = x\), and \(g(x) = x_0\).

\(\Rightarrow\): \(X\) contractible means there exists \(x_0 \in X\) such that \(1_X \simeq x_0\), i.e. there exists a homotopy \(H: X \times I \to X\) such that \(H_0(x) = x\) and \(H_1(x) = x_0\). Define \(G: Y \times I \to X\) by

\[
G_t(y) = \begin{cases} 
H_{2t}(f(y)) & \text{for } 0 \leq t < \frac{1}{2} \\
H_{2-2t}(g(y)) & \text{for } \frac{1}{2} \leq t \leq 1
\end{cases}
\]

Since \(H_1(f(y)) = H_1(g(y)) = x_0\), \(G\) is continuous.

Since \(G_0(y) = H_0(f(y)) = f(y)\) and \(G_1(y) = H_0(g(y)) = g(y)\), \(f \simeq g\).

3.7 \(X\) is simply connected \(\iff\) any \(f: S^1 \to X\) extends to the unit disk \(E^2\).

Define \(\varepsilon: I \to S^1\) by \(\varepsilon(s) = e^{2\pi i s}\). This a quotient map with \(0 \sim 1\).

\(\Rightarrow\): Given \(f\) as above, \(\sigma \overset{\text{def}}{=} f \circ \varepsilon\) is a path in \(X\).

Note that \(\sigma\) is a loop at \(x_0 \overset{\text{def}}{=} f(1)\), since \(\sigma(0) = f(\varepsilon(0)) = f(1) = f(\varepsilon(1)) = \sigma(1)\).

Since \(X\) is simply connected, \(\sigma \simeq x_0\), i.e. there exists a homotopy \(H: I \times I \to X\) such that \(H_0(s) = x_0\), \(H_1(s) = \sigma(s)\) and \(H_t(0) = H_t(1) = x_0\).

Define \(F: E^2 \to X\) as follows. If \(z \in E^2\), write it in polar form \(z = te^{2\pi i s}\) and define \(F(z) = H_t(s)\). Since polar representation is not unique we must check that \(F\) is well defined. If \(z = 0\), \(F(z)\) should be independent of \(s\) and if \(z\) is positive real, \(F(z)\) should be the same if we choose \(s = 0\) or \(s = 1\). This is indeed the case, since \(F(0) = H_0(s) = x_0\) and \(H_t(0) = H_t(1)\).

\(F\) is clearly continuous and \(F|_{S^1} \equiv f\), since for \(z \in S^1\), \(z = \varepsilon(s)\), so \(F(z) = H_1(s) = \sigma(s) = f(\varepsilon(s)) = f(z)\).

\(\Leftarrow\): If \(\sigma: I \to X\) is a loop at \(x_0\), define \(f: S^1 \to X\) by \(f(z) = \sigma(s)\). This is well defined, since \(\sigma(0) = \sigma(1)\).

Extend \(f\) to \(F: E^2 \to X\) and define \(H: I \times I \to X\) by \(H_t(s) = F(1 - t + te^{2\pi i s})\).

\(H\) is clearly continuous, \(H_0(s) = F(1) = f(1) = \sigma(0) = \sigma(1) = x_0\), \(H_1(s) = F(e^{2\pi i s}) = f(e^{2\pi i s}) = \sigma(s)\), and \(H_t(0) = H_t(1) = F(1) = x_0\). Thus, \(H\) is the required homotopy for \(\sigma \simeq x_0\).