5 Vector Functions

5.1 Limit and Continuity

The students are familiar with properties of real valued functions. Now we will define functions whose domain is the set of all real numbers and the range is a set of vectors. Since in many applications we will use three-dimensional vectors we give the following definition.

Definition 5.1 If corresponding to the real number t a function assigns uniquely determined vector $\overrightarrow{r} =< x, y, z >$ in three-dimensional space, then the function

$$\overrightarrow{r}(t) = x(t)\overrightarrow{i} + y(t)\overrightarrow{j} + z(t)\overrightarrow{k} \tag{5.1}$$

is called a *vector function in R.*

It is clear that the components x, y and z of a vector function $\overrightarrow{r}$ are real-valued scalar functions $x = x(t)$, $y = y(t)$ and $z = z(t)$ and we will called them the *components functions* of $\overrightarrow{r}$.

We can write vector functions as $\overrightarrow{r}(t) = < x(t),\ y(t),\ z(t) >$ and they satisfy all Laws of the Vector Algebra. Also, the components of a vector function satisfy all properties of the real valued functions.

Definition 5.2 If the limits of all components of vector functions exist, then the *limit* of a vector function $\overrightarrow{r}(t) = < x(t),\ y(t),\ z(t) >$ is

$$\lim_{t\to t_0} \overrightarrow{r}(t) = < \lim_{t\to t_0} x(t),\ \lim_{t\to t_0} y(t),\ \lim_{t\to t_0} z(t) > .$$

Definition 5.2 shows that the limit of a vector function can be obtained by taking the limits of its components . It is easy to verify that if the limits $\lim\limits_{t\to t_0} \overrightarrow{r}_1(t)$, $\lim\limits_{t\to t_0} \overrightarrow{r}_2(t)$ and $\lim\limits_{t\to t_0} f(t)$ exist , where $f(t)$ is a scalar function, then

1. $\lim\limits_{t\to t_0}[\overrightarrow{r}_1(t) + \overrightarrow{r}_2(t)] = \lim\limits_{t\to t_0} \overrightarrow{r}_1(t) + \lim\limits_{t\to t_0} \overrightarrow{r}_2(t).$
2. $\lim\limits_{t\to t_0}[f(t)\overrightarrow{r}_1(t)] = \lim\limits_{t\to t_0} f(t) \lim\limits_{t\to t_0} \overrightarrow{r}_1(t).$
3. $\lim\limits_{t\to t_0}[\overrightarrow{r}_1(t).\overrightarrow{r}_2(t)] = \lim\limits_{t\to t_0} \overrightarrow{r}_1(t). \lim\limits_{t\to t_0} \overrightarrow{r}_2(t).$

4. $\lim\limits_{t\to t_0}[\vec{r}_1(t) \times \vec{r}_2(t)] = \lim\limits_{t\to t_0} \vec{r}_1(t) \times \lim\limits_{t\to t_0} \vec{r}_2(t)$.

Example 1. The limit $\lim\limits_{t\to 0} \vec{r}(t)$ of the function $\vec{r}(t) = (\sqrt{t} - 3)\vec{i} + tan\, t\, \vec{j} + \dfrac{sin\, 5t}{sin\, t}\vec{k}$ is

$$\lim_{t\to 0} \vec{r}(t) = \left[\lim_{t\to 0}(\sqrt{t} - 3)\right]\vec{i} + \left[\lim_{t\to 0}(tan\, t)\right]\vec{j} + \left[\lim_{t\to 0}\frac{sin\, 5t}{sin\, t}\right]\vec{k}$$

$$= -3\vec{i} + 5\vec{k}.$$

Example 2. Let $\vec{r}_1(t) = (1+t)\vec{i} - (t^2-1)\vec{j} + (2t+5)\vec{k}$ and $\vec{r}_2(t) = t^3\vec{i} + \frac{1}{t}\vec{j} + 2t\vec{k}$. Using Property 3 we obtain

$$\lim_{t\to 1}[\vec{r}_1(t).\vec{r}_2(t)] = \lim_{t\to 1} \vec{r}_1(t). \lim_{t\to 1} \vec{r}_2(t)$$

$$= (2\vec{i} + 7\vec{k}).(\vec{i} + \vec{j} + 2\vec{k}) = 16.$$

Definition 5.3 A vector function $\vec{r}(t) = < x(t),\ y(t),\ z(t) >$ is *continuous* at a point t_0 if

$$\lim_{t\to t_0} \vec{r}(t) = \vec{r}(t_0) = < x(t_0),\ y(t_0),\ z(t_0) > .$$

According to Definition 5.2 it is easy to verify that $\vec{r}(t)$ is continuous at t_0 if and only if all its components are continuous at t_0.

Definition 5.4 The *graph* of a continuous vector function $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ for $t \in [a, b]$ is the set of all points $M(x(t), y(t), z(t))$ with position vector $\vec{r}(t) = < x(t),\ y(t),\ z(t) >$. (Fig. 5.1)

5.1

We can see from the Fig. 5.1 that the graph of a continuous vector function $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ for $t \in [a,\ b]$ is a three-dimensional *curve* (c) or a *space curve*. Each point $M(x,\ y,\ z)$ on (c) has coordinates

$$(c): \ x = x(t), \quad y = y(t), \quad z = z(t), \quad t \in [a,\ b] \qquad (5.2)$$

The equations (5.2) are called *parametric equations* of (c) and t is called a *parameter*. When the parameter t takes all values in the interval $[a, b]$ the vector $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ shows the position of each point on (c). Therefore, all space curves can be represented as graphs of vector functions. In particular case when the vector function $\vec{r}(t)$ has two components we can represent the plane curves.

Example 3. For the line l through the point $M_0(4,\, 5,\, 2)$ and parallel to the vector $\vec{a} = <-1,\, 3,\, -1>$ a vector parametric equation is

$$(l):\ \vec{r} = 4\vec{i} + 5\vec{j} + 2\vec{k} + \lambda(-\vec{i} + 3\vec{j} - \vec{k})$$

or

$$(l):\ \vec{r} = (4-\lambda)\vec{i} + (5+3\lambda)\vec{j} + (2-\lambda)\vec{k}.$$

Thus the line (l) can be represents as a graph of the vector function $\vec{r} = (4-t)\vec{i} + (5+3t)\vec{j} + (2-t)\vec{k}$, where $t \in R$.

Example 4. The curve in the xy-plane with parametric equations

$$(c):\ x = R\,cos\,t, \quad y = R\,\sin\,t, \quad t \in [0,\, 2\pi], \quad R > 0$$

is a circle centered at the origin $O(0,\, 0)$ with radius R. The parameter t represent the angle between the positive $x-$axis and the position vector of a point in the circle.(Fig. 5.2)

Fig. 5.2

5.2 Derivatives and Integrals

In the previous section we defined the limit of a vector function by the limits of all its components. Similarly, we can define the derivatives.

Definition 5.5 The *derivative* of a vector function $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ at a point $t = t_0$, is defined as $\vec{r}\,'(t_0) = \lim_{h\to 0}\frac{\vec{r}(t_0+h)-\vec{r}(t_0)}{h}$ if the limit exists; otherwise, $\vec{r}(t)$ does not have a derivative at $t = t_0$. In general, at t,

$$\vec{r}\,'(t) = \lim_{h\to 0}\frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

When a vector function has a derivative at a point, we say it is *differentiable* at that point. A function $\vec{r}(t)$ is said to be differentiable on an interval $I = (a, b)$ if it is differentiable at every point t in this interval. The notation $\frac{d\overrightarrow{r}(t)}{dt}$ is also used for the derivative of a function at t, i.e. $\frac{d\overrightarrow{r}(t)}{dt} = \overrightarrow{r}'(t)$.

Since

$$\frac{\overrightarrow{r}(t+h) - \overrightarrow{r}(t)}{h}$$

$$= \frac{x(t+h) - x(t)}{h}\overrightarrow{i} + \frac{y(t+h) - y(t)}{h}\overrightarrow{j} + \frac{z(t+h) - z(t)}{h}\overrightarrow{k},$$

then it is clear that *a vector function is differentiable if and only if its components are differentiable* and

$$\overrightarrow{r}'(t) = x'(t)\overrightarrow{i} + y'(t)\overrightarrow{j} + z'(t)\overrightarrow{k}. \tag{5.3}$$

The equality (5.3) gives a convenient method to calculate the derivative of a vector function; just differentiate each its components.

Example 1. The derivative of the vector function $\overrightarrow{r}(t) = (2 + t^5)\overrightarrow{i} + \sqrt{t}\overrightarrow{j} + cos\ 3t\overrightarrow{k}$ is

$$\overrightarrow{r}'(t) = 5t^4\overrightarrow{i} + \frac{1}{2\sqrt{t}}\overrightarrow{j} - 3\ sin\ 3t\overrightarrow{k}.$$

The following theorem can be proved easily from Definition 5.5.

Theorem 5.1*Assuming that $\overrightarrow{r}_1(t)$ and $\overrightarrow{r}_2(t)$ are differentiable vector functions, then*

1. $[k\overrightarrow{r}_1(t)]' = k\overrightarrow{r}'_1(t)$ *for any constant* k.
2. $[\overrightarrow{r}_1(t) + \overrightarrow{r}_2(t)]' = \overrightarrow{r}'_1(t) + \overrightarrow{r}'_2(t)$.
3. $[f(t).\overrightarrow{r}_1(t)]' = f'(t)\overrightarrow{r}_1(t) + \overrightarrow{r}'_1(t)f(t)$

   for any differentiable scalar function $f(t)$.
4. $[\overrightarrow{r}_1(t).\overrightarrow{r}_2(t)]' = \overrightarrow{r}'_1(t).\overrightarrow{r}_2(t) + \overrightarrow{r}_1(t).\overrightarrow{r}'_2(t)$.
5. $[\overrightarrow{r}_1(t) \times \overrightarrow{r}_2(t)]' = \overrightarrow{r}'_1(t) \times \overrightarrow{r}_2(t) + \overrightarrow{r}_1(t) \times \overrightarrow{r}'_2(t)$.
6. $[\overrightarrow{r}_1(f(t))]' = f'(t)\overrightarrow{r}'_1(f(t))$

   for any differentiable scalar function $f(t)$. *(Chain rule)*

Example 2. The derivatives of the functions $\vec{r}_1(t) = t^2\vec{i} - \vec{j} + 3t\vec{k}$ and $\vec{r}_2(t) = -4t\vec{i} + sin\,t\,\vec{j} - t^3\vec{k}$ are $\vec{r}'_1(t) = 2t\vec{i} + 3\vec{k}$ and $\vec{r}'_2(t) = -4\vec{i} + cos\,t\,\vec{j} - 3t^2\vec{k}$. Hence the derivative of their cross product $\vec{r}_1(t) \times \vec{r}_2(t)$ is

$$\frac{d}{dt}[\vec{r}_1(t) \times \vec{r}_2(t)] = \vec{r}'_1(t) \times \vec{r}_2(t) + \vec{r}_1(t) \times \vec{r}'_2(t)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & 0 & 3 \\ -4t & sin\,t & -t^3 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t^2 & -1 & 3t \\ -4 & cos\,t & -3t^2 \end{vmatrix}$$

$$= -3\,sin\,t\,\vec{i} + (2t^4 - 12t)\vec{j} + (2t\,sin\,t)\vec{k}$$

$$+(3t^2 - 3t\,cos\,t)\vec{i} + (3t^4 - 12t)\vec{j} + (t^2\,cos\,t - 4)\vec{k}$$

$$= (3t^2 - 3t\,cos\,t - 3\,sin\,t)\vec{i} + (5t^4 - 24t)\vec{j} + (t^2\,cos\,t + 2t\,sin\,t - 4)\vec{k}\,.$$

Notice that the same result can be obtained by calculating the cross product $\vec{r}_1(t) \times \vec{r}_2(t)$ and then differentiating its components.

Just as for real functions we can defined the higher order derivatives of vector functions, i.e. $\vec{r}''(t) = \vec{r}'(t)$, $\vec{r}'''(t) = \vec{r}''(t)$, etc.

Example 3. The second derivative of the vector function $\vec{r}(t) = t^7\vec{i} - e^{5t}\vec{j} + 2\,cos\,t\,\vec{k}$ is

$$\vec{r}''(t) = 42t^5\vec{i} - 25e^{5t}\vec{j} - 2\,cos\,t\,\vec{k}\,.$$

Again using components we give the following definitions of indefinite and definite integral of vector functions.

Definition 5.6 The vector function $\vec{F}(t) = X(t)\vec{i} + Y(t)\vec{j} + Z(t)\vec{k}$ is an *antiderivative* for the vector function $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ on the interval I if $\vec{F}'(t) = \vec{r}(t)$ for all $t \in I$. The set of all antiderivatives of $\vec{r}(t)$ on I is called the *indefinite integral* of $\vec{r}(t)$ on I.

From Definition 5.6 we can obtain the following

Theorem 5.2 *Assuming that the integrals of the components on I exist, then*

$$\int \overrightarrow{r}(t)\,dt = \Big[\int x(t)\,dt\Big]\overrightarrow{i} + \Big[\int y(t)\,dt\Big]\overrightarrow{j} + \Big[\int z(t)\,dt\Big]\overrightarrow{k}.$$

The definite integral of a vector function is defined as follows

Definition 5.7 If the definite integrals of the components exist, then the *definite integral* of the vector function $\overrightarrow{r}(t) = x(t)\,\overrightarrow{i} + y(t)\,\overrightarrow{j} + z(t)\,\overrightarrow{k}$ on the interval $[a,\ b]$ is

$$\int_a^b \overrightarrow{r}(t)\,dt = \Big[\int_a^b x(t)\,dt\Big]\overrightarrow{i} + \Big[\int_a^b y(t)\,dt\Big]\overrightarrow{j} + \Big[\int_a^b z(t)\,dt\Big]\overrightarrow{k}.$$

Example 4. For the vector function $\overrightarrow{r}(t) = t\,\overrightarrow{i} - 4t^3\,\overrightarrow{j} + 3^t\,\overrightarrow{k}$ we have that

$$\begin{aligned}\int \overrightarrow{r}(t)\,dt &= \Big[\int t\,dt\Big]\overrightarrow{i} + \Big[-4\int t^3\,dt\Big]\overrightarrow{j} + \Big[\int 3^t\,dt\Big]\overrightarrow{k}\\ &= \Big[\frac{t^2}{2} + c_1\Big]\overrightarrow{i} + \Big[-t^4 + c_2\Big]\overrightarrow{j} + \Big[\frac{3^t}{ln\,3} + c_3\Big]\overrightarrow{k}\\ &= \frac{t^2}{2}\overrightarrow{i} - t^4\overrightarrow{j} + \frac{3^t}{ln\,3}\overrightarrow{k} + \overrightarrow{C},\end{aligned}$$

where $\overrightarrow{C} = c_1\,\overrightarrow{i} + c_2\,\overrightarrow{j} + c_3\,\overrightarrow{k}$.

Example 5. For the vector function $\overrightarrow{r}(t) = sin\,t\,\overrightarrow{i} + 2\,cos\,t\,\overrightarrow{j} - 4\,\overrightarrow{k}$ we have that

$$\begin{aligned}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \overrightarrow{r}(t)\,dt &= \Big[\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} sin\,t\,dt\Big]\overrightarrow{i} + \Big[2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} cos\,t\,dt\Big]\overrightarrow{j} + \Big[\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (-4)\,dt\Big]\overrightarrow{k}\\ &= \Big[-cos\,t\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}\Big]\overrightarrow{i} + \Big[2\,sin\,t\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}\Big]\overrightarrow{j} + \Big[-4t\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}\Big]\overrightarrow{k}\\ &= \Big[0 + \frac{\sqrt{2}}{2}\Big]\overrightarrow{i} + \Big[2.1 - 2.\frac{\sqrt{2}}{2}\Big]\overrightarrow{j} + \Big[-4\frac{\pi}{2} + 4\frac{\pi}{4}\Big]\overrightarrow{k}\\ &= \frac{\sqrt{2}}{2}\overrightarrow{i} + (2 - \sqrt{2})\,\overrightarrow{j} - \pi\,\overrightarrow{k}.\end{aligned}$$

5.3 Applications

The Tangent Vector and The Tangent Line

As we know the graph of a *continuous* vector function $\overrightarrow{r}(t) = x(t)\overrightarrow{i} + y(t)\overrightarrow{j} + z(t)\overrightarrow{k}$ for $t \in [a,\ b]$ is a three-dimensional curve (c). If the derivative $\overrightarrow{r}'(t)$ of the vector function is also continuous function and $\overrightarrow{r}'(t) \neq 0$ for $t \in (a,\ b)$, then the curve given by $\overrightarrow{r}(t)$ is called *smooth* on $[a,\ b]$.

Let the curve (c) determined by the vector function $\overrightarrow{r}(t)$ is smooth in a neighborhood of a point $t_0 \in (a,\ b)$. Let $\overrightarrow{r}(t_0)$ and $\overrightarrow{r}(t_0+h)$, $h > 0$ are the position vectors of the points M_0 and M on (c). Then the line through M_0 and M is a *secant line* to (c). (Fig. 5.3) The vector $\frac{\overrightarrow{r}(t_0+h)-\overrightarrow{r}(t_0)}{h}$ is parallel to the secant line and is called a *secant vector*. Now, if we let h go to 0, the point M approaches the point M_0 , the secant line approaches the *line tangent* to the curve at M_0 and the secant vector $\frac{\overrightarrow{r}(t_0+h)-\overrightarrow{r}(t_0)}{h}$ approaches the *tangent vector* to the curve at M_0. Therefore, $\lim_{h\to 0} \frac{\overrightarrow{r}(t_0+h)-\overrightarrow{r}(t_0)}{h}$=the tangent vector at M_0. Thus,

$$\lim_{h\to 0} \frac{\overrightarrow{r}(t_0+h)-\overrightarrow{r}(t_0)}{h} = \overrightarrow{r}'(t_0)$$

if the curve (c) determined by the vector function $\overrightarrow{r}(t)$ is smooth in $[a,\ b]$.

Fig. 5.3

Since the tangent line to (c) at M_0 is determined by the point $M_0(x(t_0), y(t_0), z(t_0))$ and the tangent vector $\overrightarrow{r}'(t_0)$ at $M_0(x(t_0),\ y(t_0),\ z(t_0))$, then it has the following vector parametric equation:

$$\overrightarrow{\rho} = \overrightarrow{r}(t_0) + \lambda \overrightarrow{r}'(t_0), \qquad \lambda \in R,$$

where $\overrightarrow{\rho}$ is the position vector of a point on the tangent line.

The *scalar parametric equations of the tangent line* to (c) at $M_0(x(t_0), y(t_0), z(t_0))$ are

$$x = x(t_0) + \lambda x'(t_0), \qquad y = y(t_0) + \lambda y'(t_0), \qquad z = z(t_0) + \lambda z'(t_0),$$

where $x,\ y,\ z$ are coordinates of a point on the tangent line.

We can also determine the *unit tangent vector* at any point $M(x(t), y(t), z(t))$ on (c), which is

$$\overrightarrow{T}(t) = \frac{\overrightarrow{r}'(t)}{|\overrightarrow{r}'(t)|}, \qquad t \in [a,\ b].$$

Example 1. Consider the vector function $\overrightarrow{r}(t) = (t+1)\overrightarrow{i} + \sqrt{3} sin t\overrightarrow{j} + (2 + cos\ t)\overrightarrow{k}$.

(a) The derivative $\overrightarrow{r}'(t) = x'(t)\overrightarrow{i} + y'(t)\overrightarrow{j} + z'(t)\overrightarrow{k}$ is

$$\overrightarrow{r}'(t) = \overrightarrow{i} + \sqrt{3}\, cos\, t\, \overrightarrow{j} - sin\, t\, \overrightarrow{k}.$$

For $t = 0$ the position vector is $\overrightarrow{r}(0) = (0+1)\overrightarrow{i} + \sqrt{3}\, sin\, 0\, \overrightarrow{j} + (2 + cos\ 0)\overrightarrow{k}$ and it determine the point $M(x(0),\ y(0),\ z(0)) = M(1,\ 0,\ 3)$ on (c). Hence the tangent vector at the point $M(1,\ 0,\ 3)$ is

$$\overrightarrow{r}'(0) = x'(0)\overrightarrow{i} + y'(0)\overrightarrow{j} + z'(0)\overrightarrow{k} = \overrightarrow{i} + \sqrt{3}\overrightarrow{j}.$$

(b) Since

$$|\overrightarrow{r}'(0)| = \sqrt{1^2 + (\sqrt{3})^2} = 2,$$

then the unit tangent vector at the point $M(1,\ 0,\ 3)$ is

$$\overrightarrow{T}(0) = \frac{\overrightarrow{r}'(0)}{|\overrightarrow{r}'(0)|} = \frac{\overrightarrow{i} + \sqrt{3}\overrightarrow{j}}{2} = \frac{1}{2}\overrightarrow{i} + \frac{\sqrt{3}}{2}\overrightarrow{j}.$$

(c) The scalar parametric equations for the tangent line at $M(1,\ 0,\ 3)$ are

$$x = x(0) + \lambda x'(0), \qquad y = y(0) + \lambda y'(0), \qquad z = z(0) + \lambda z'(0),$$

or

$$x = 1 + \lambda\frac{1}{2}, \qquad y = \lambda\frac{\sqrt{3}}{2}, \qquad z = 3.$$

The Normal and Binormal Vectors

Let the graph of the vector function $\overrightarrow{r}(t) = x(t)\overrightarrow{i} + y(t)\overrightarrow{j} + z(t)\overrightarrow{k}$ for $t \in [a,\ b]$ is the smooth space curve (c). Let $\overrightarrow{T}(t)$ is the unit tangent vector at $t \in [a,\ b]$. (Fig. 5.4) There exist many vectors which are orthogonal to $\overrightarrow{T}(t)$. Now we will consider one more special such a vector.

Since $|\overrightarrow{T}(t)| = 1$ by the definition of the scalar product we have that

$$\overrightarrow{T}(t).\overrightarrow{T}(t) = 1.$$

Hence

$$0 = \frac{d}{dt}[\overrightarrow{T}(t).\overrightarrow{T}(t)] = \overrightarrow{T}'(t).\overrightarrow{T}(t) + \overrightarrow{T}(t).\overrightarrow{T}'(t) = 2\overrightarrow{T}'(t).\overrightarrow{T}(t)$$

by the Property 4 of the derivatives of a vector function. Therefore the vector $\overrightarrow{T}'(t)$ is orthogonal to $\overrightarrow{T}(t)$ for all $t \in [a,\ b]$. The vector $\overrightarrow{T}'(t)$ is called the *normal vector* to the curve at t. It may or may not be unit. If the curve given by $\overrightarrow{r}'(t)$ is also smooth we may define the *unit normal vector* $\overrightarrow{N}(t)$ as follows:

$$\overrightarrow{N}(t) = \frac{\overrightarrow{T}'(t)}{|\overrightarrow{T}'(t)|}, \qquad t \in [a,\ b]$$

The vector $\overrightarrow{B}(t) = \overrightarrow{T}(t) \times \overrightarrow{N}(t)$, $t \in [a,\ b]$ is called the *binormal vector* at t. According to the definition of the cross product it is orthogonal to both $\overrightarrow{T}(t)$ and $\overrightarrow{N}(t)$ and is also an unit vector.

Fig. 5.4

Example 2. For the curve given by the vector function $\overrightarrow{r}(t) = e^t sint\,\overrightarrow{i} + e^t\, cos\, t\, \overrightarrow{j} + e^t\overrightarrow{k}$, $t \in [0, 2\pi]$ we have that

$$\overrightarrow{r}'(t) = e^t(sin\, t + cos\, t)\,\overrightarrow{i} + e^t(cos\, t - sin\, t)\,\overrightarrow{j} + e^t\,\overrightarrow{k}$$

and hence

$$|\overrightarrow{r}'(t)| = e^t\sqrt{(sin\, t + cos\, t)^2 + (cos\, t - sin\, t)^2 + 1} = \sqrt{3}e^t.$$

Therefore, the unit tangent vector is defined for all $t \in [0, 2\pi]$ by

$$\overrightarrow{T}(t) = \frac{\overrightarrow{r}'(t)}{|\overrightarrow{r}'(t)|} = \frac{e^t(sin\, t + cos\, t)\,\overrightarrow{i} + e^t(cos\, t - sin\, t)\,\overrightarrow{j} + e^t\,\overrightarrow{k}}{\sqrt{3}e^t}$$

$$= \frac{sin\,t + cos\,t}{\sqrt{3}}\overrightarrow{i} + \frac{cos\,t - sin\,t}{\sqrt{3}}\overrightarrow{j} + \frac{1}{\sqrt{3}}\overrightarrow{k}.$$

Also,

$$\overrightarrow{T}'(t) = \frac{cos\,t - sin\,t}{\sqrt{3}}\overrightarrow{i} - \frac{sin\,t + cos\,t}{\sqrt{3}}\overrightarrow{j}$$

and

$$|\overrightarrow{T}'(t)| = \frac{1}{\sqrt{3}}\sqrt{(cos\,t - sin\,t)^2 + (sin\,t + cos\,t)^2} = \sqrt{\frac{2}{3}}.$$

Thus, the unit normal vector is

$$\overrightarrow{N}(t) = \frac{\overrightarrow{T}'(t)}{|\overrightarrow{T}'(t)|}$$

$$= \frac{cos\,t - sin\,t}{\sqrt{2}}\overrightarrow{i} - \frac{sin\,t + cos\,t}{\sqrt{2}}\overrightarrow{j}, \quad t \in [0, 2\pi]$$

and the binormal vector is

$$\overrightarrow{B}(t) = \overrightarrow{T}(t) \times \overrightarrow{N}(t) = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{sin\,t+cos\,t}{\sqrt{3}} & \frac{cos\,t-sin\,t}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{cos\,t-sin\,t}{\sqrt{2}} & -\frac{sin\,t+cos\,t}{\sqrt{2}} & 0 \end{vmatrix}$$

$$= \frac{sin\,t + cos\,t}{\sqrt{6}}\overrightarrow{i} + \frac{cos\,t - sin\,t}{\sqrt{6}}\overrightarrow{j} - \frac{2}{\sqrt{6}}\overrightarrow{k}.$$

Let (c) is the curve given by the vector function $\overrightarrow{r}(t) = x(t)\overrightarrow{i} + y(t)\overrightarrow{j} + z(t)\overrightarrow{k}$ for $t \in I$. Let $M(x(t), y(t), z(t))$ is an arbitrary point on (c). The plane through M determined by the vectors $\overrightarrow{N}(t)$ and $\overrightarrow{B}(t)$ is called the *normal plane* of (c) at M. It is clear that this plane consists of all lines that are orthogonal to the tangent vector of (c) at M. The plane through M determined by the vectors $\overrightarrow{T}(t)$ and $\overrightarrow{N}(t)$ is called the *osculating plane* of (c) at M. Notice that for the plane curve, the osculating plane is the plane that contains the curve.

Example 3. For the curve determined by the vector function from the Example 2 the unit tangent, the unit normal and the binormal vectors at $t = 0$ are

$$\overrightarrow{T}(0) = \frac{1}{\sqrt{3}}\overrightarrow{i} + \frac{1}{\sqrt{3}}\overrightarrow{j} + \frac{1}{\sqrt{3}}\overrightarrow{k},$$

$$\overrightarrow{N}(0) = \frac{1}{\sqrt{2}}\overrightarrow{i} - \frac{1}{\sqrt{2}}\overrightarrow{j},$$

$$\overrightarrow{B}(0) = \frac{1}{\sqrt{6}}\overrightarrow{i} + \frac{1}{\sqrt{6}}\overrightarrow{j} - \frac{2}{\sqrt{6}}\overrightarrow{k}.$$

Also, the coordinates of the point $M(x(t),\, y(t),\, z(t))$ on (c) for $t = 0$ are the components of the position vector

$$\overrightarrow{r}(0) = <0,\ 1,\ 1> .$$

The normal plane of (c) at $M(0,\, 1,\, 1)$ has a normal vector $\overrightarrow{T}(0)$ and hence it has the standard equation

$$\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}(y-1) + \frac{1}{\sqrt{3}}(z-1) = 0$$

or

$$x + y + z - 2 = 0.$$

The osculating plane of (c) at $M(0,\, 1,\, 1)$ has a normal vector $\overrightarrow{B}(0)$ and hence it has the standard equation

$$\frac{1}{\sqrt{6}}x + \frac{1}{\sqrt{6}}(y-1) - \frac{2}{\sqrt{6}}(z-1) = 0$$

or

$$x + y - 2z + 1 = 0.$$

Arc Length and Curvature

A curve that is made up of a finite number of smooth pieces is called *piecewise smooth.*

Let (c) is a piecewise smooth curve given by the vector function $\overrightarrow{r}(t) = x(t)\overrightarrow{i} + y(t)\overrightarrow{j} + z(t)\overrightarrow{k}$ for $t \in [a,\, b]$ and (c) is traversed exactly once as t increases from a to b. The *length* s of curve (c) can be calculate by

$$s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} dt. \tag{5.4}$$

The equality (5.4) can be proved in the same way as the formula for the length of a plane curve. (Calculus II) Since the vector function

$\overrightarrow{r}'(t)$ has components $x'(t)$, $y'(t)$ and $z'(t)$, then the equality (5.4) can be written in a more compact form as

$$s = \int_a^b |\overrightarrow{r}'(t)|dt.$$

In case when the curve (c) is a plane curve we obtain

$$s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}dt. \tag{5.5}$$

for the length of (c).

Example 4. For the length of the plane curve given by the vector function $\overrightarrow{r}(t) = a(t - sin\ t)\overrightarrow{i} + a(1 - cos\ t)\overrightarrow{j}$ for $t \in [0, 2\pi]$ $(a > 0)$ we will use (5.5). We have that

$$s = \int_0^{2\pi} \sqrt{[x'(t)]^2 + [y'(t)]^2}dt = \int_0^{2\pi} a\sqrt{2(1 - cos\ t)}dt$$

$$= 2a\int_0^{2\pi} sin\frac{t}{2}dt = -4a\ cos\frac{t}{2}\Big|_0^{2\pi} = 8a.$$

Notice that a curve can be represent by more than one vector function. Also, we can use different parameters for the representation of a curve. We say that these are different *parametrizations* of the curve (c). For example, if a plane curve (c) is given as a graph of a function $y = f(x)$, $x \in [a, b]$ we can parametrize the curve by

$$x = t, \quad y = f(t),$$

and from (5.5) we obtain the known formula

$$s = \int_a^b \sqrt{1 + [y'(x)]^2}dx.$$

If in (5.4) the upper bound of the integral is a variable t, $t \in [a, b]$ we can define the *arc length function* $s = s(t)$ by

$$s(t) = \int_a^t |\overrightarrow{r}'(u)|du = \int_a^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}du. \tag{5.6}$$

Thus, $s(t)$ is the length of the part of (c) between the points with position vectors $\overrightarrow{r}(a)$ and $\overrightarrow{r}(t)$ (Fig. 5.5)

Fig. 5.5

If we differentiate both side of (5.6) using the Fundamental Theorem of Calculus, we obtain

$$\frac{ds}{dt} = |\overrightarrow{r}'(t)|. \tag{5.7}$$

Since $\frac{ds}{dt} = |\overrightarrow{r}'(t)| \geq 0$ for $t \in [a,\ b]$, the function $s = s(t)$ is monotone increasing for $t \in [a,\ b]$ and, hence, has an inverse $t = t(s)$. If we substitute into the vector function $\overrightarrow{r}(t)$ parameter t with $t = t(s)$, we obtain

$$\overrightarrow{r}(t) = \overrightarrow{r}(t(s)) = \overrightarrow{r}(s). \tag{5.8}$$

The equation (5.8) is called the vector parametric equation of (c) *with respect to arc length.* It is easily to verify that the following Theorem holds.

Theorem 5.3 *If a curve* (c) *is parametrized by arc length then the unit tangent vector* $\overrightarrow{T}(s)$ *is*

$$\overrightarrow{T}(s) = \overrightarrow{r}'(s).$$

Proof. We use the Chain Rule to write

$$\overrightarrow{r}'(s) = \frac{dr}{ds} = \frac{dr}{dt}.\frac{dt}{ds} = \frac{dr}{dt}.\frac{1}{\frac{ds}{dt}}$$

From (5.7) we have

$$\overrightarrow{r}'(s) = \frac{\overrightarrow{r}'(t)}{|\overrightarrow{r}'(t)|} = \overrightarrow{T}(s).$$

Theorem 5.3 show that if a curve is parametrized by arc length, then *the length of its derivative is constantly equal to 1.* Because of that it is often useful to parametrize a curve by arc length.

Definition 5.8 The *curvature* of a curve is

$$\kappa = \left|\frac{dT}{ds}\right|,$$

where $\overrightarrow{T}$ is the unit tangent vector.

Using (5.7) we obtain

$$\kappa(t) = \left|\frac{dT}{ds}\right| = \left|\frac{dT/dt}{ds/dt}\right| = \frac{|\overrightarrow{T}'(t)|}{|\overrightarrow{r}'(t)|}. \tag{5.9}$$

From (5.9) we can obtain

$$\kappa(t) = \frac{|\overrightarrow{r}'(t) \times \overrightarrow{r}''(t)|}{|\overrightarrow{r}'(t)|^3}.$$

Let the curve (c) is given by the vector function $\overrightarrow{r}(t)$ and has curvature $\kappa = \kappa(t)$ at a point $M(x(t),\, y(t),\, z(t))$. The expression $\rho = \frac{1}{\kappa}$ is called *radius of curvature at M*. The circle that lies in the osculating plane of (c) at M, has the same tangent as (c) at M, lies on the concave side of (c), and has a radius $R = \rho = \frac{1}{\kappa}$ is called *osculating circle* of (c) at M. It has the same tangent, normal and curvature at M.

Example 5. For the curve given by the vector function $\overrightarrow{r}(t) = t^2\overrightarrow{i} + t^3\overrightarrow{j} + t\,\overrightarrow{k}$, we have that

$$\overrightarrow{r}'(t) = 2t\,\overrightarrow{i} + 3t^2\overrightarrow{j} + \overrightarrow{k}$$

and

$$|\overrightarrow{r}'(t)| = \sqrt{4t^2 + 9t^4 + 1}.$$

Also,

$$\overrightarrow{r}''(t) = 2\,\overrightarrow{i} + 6t\,\overrightarrow{j}$$

and hence,

$$\overrightarrow{r}'(t) \times \overrightarrow{r}''(t) = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 2t & 3t^2 & 1 \\ 2 & 6t & 0 \end{vmatrix} = -6t\,\overrightarrow{i} + 2\,\overrightarrow{j} + 6t^2\overrightarrow{k}.$$

Therefore,

$$\kappa(t) = \frac{|\overrightarrow{r}'(t) \times \overrightarrow{r}''(t)|}{|\overrightarrow{r}'(t)|^3} = \frac{\sqrt{36t^2 + 4 + 36t^4}}{(4t^2 + 9t^4 + 1)^{3/2}}.$$

The curvature of the curve at $M(1,\ 1,\ 1)$ can be obtained for $t = 1$

$$\kappa(1) = \frac{\sqrt{19}}{7\sqrt{14}}$$

and the radius of curvature at $M(1,\ 1,\ 1)$ is

$$\rho(1) = \frac{1}{\kappa(1)} = \frac{7\sqrt{14}}{\sqrt{19}}.$$

The curvature of the curve at the origin $O(0,\ 0,\ 0)$ can be obtained for $t = 0$

$$\kappa(0) = 2$$

and the radius of curvature at $O(0,\ 0,\ 0)$ is

$$\rho(0) = \frac{1}{\kappa(0)} = \frac{1}{2}.$$

Velocity and Acceleration

Let $\overrightarrow{r}(t) = x(t)\overrightarrow{i} + y(t)\overrightarrow{j} + z(t)\overrightarrow{k}$ be a differentiable vector function on I. The location of a moving object at time t_0 can be given by the point $M(x(t_0),\ y(t_0),\ z(t_0))$ on the graph of the vector function $\overrightarrow{r}(t)$ described by the position vector $\overrightarrow{r}(t_0) = x(t_0)\overrightarrow{i} + y(t_0)\overrightarrow{j} + z(t_0)\overrightarrow{k}$. The *average velocity* between the times t_0 and $t_0 + h$ would be

$$\overrightarrow{v}_a(t_0) = \frac{\overrightarrow{r}(t_0 + h) - \overrightarrow{r}(t_0)}{h}.$$

As h gets close to 0, the vector $\overrightarrow{r}(t_0 + h)$ gets close to the vector $\overrightarrow{r}(t_0)$ and we can define the *instantaneous velocity* at time t_0 as

$$\overrightarrow{v}(t_0) = \lim_{h\to 0} \frac{\overrightarrow{r}(t_0 + h) - \overrightarrow{r}(t_0)}{h} = \overrightarrow{r}'(t_0).$$

Notice that velocity at t_0 is a vector $\overrightarrow{v}(t_0) = \overrightarrow{r}'(t_0) = x'(t_0)\overrightarrow{i} + y'(t_0)\overrightarrow{j} + z'(t_0)\overrightarrow{k}$. Since $\overrightarrow{r}(t)$ is differentiable at any point $t \in I$, then we obtain the velocity function for $\overrightarrow{r}(t)$ on the interval I as

$$\overrightarrow{v}(t) = \overrightarrow{r}'(t) = x'(t)\overrightarrow{i} + y'(t)\overrightarrow{j} + z'(t)\overrightarrow{k}, \quad t \in I.$$

By the *speed* of an object at t_0, we shall mean the length of the velocity vector $|\overrightarrow{v}(t_0)| = |\overrightarrow{r}'(t_0)|$.

Similarly, we define the *acceleration* of a moving object in space at the moment t_0 as rate of change of velocity with respect to time

$$\overrightarrow{a}(t_0) = \overrightarrow{v}'(t_0)$$

or

$$\overrightarrow{a}(t_0) = \overrightarrow{r}''(t_0) = x''(t_0)\overrightarrow{i} + y''(t_0)\overrightarrow{j} + z''(t_0)\overrightarrow{k}.$$

Example 6. The position vector of a moving object is given by $\overrightarrow{r}(t) = (t^2+1)\overrightarrow{i} + t^3\overrightarrow{j} + (2-t)\overrightarrow{k}$. Since

$$\overrightarrow{r}'(t) = 2t\overrightarrow{i} + 3t^2\overrightarrow{j} - \overrightarrow{k}$$

and

$$\overrightarrow{r}''(t) = 2\overrightarrow{i} + 6t\overrightarrow{j},$$

then the velocity, speed, and acceleration of the object when $t = 2$ are

$$\begin{aligned} \overrightarrow{v}(2) = \overrightarrow{r}'(2) &= 4\overrightarrow{i} + 12\overrightarrow{j} - \overrightarrow{k}, \\ |\overrightarrow{v}(2)| &= \sqrt{4^2+12^2+(-1)^2} = \sqrt{161}, \\ \overrightarrow{a}(2) = \overrightarrow{r}''(2) &= 2\overrightarrow{i} + 12\overrightarrow{j}. \end{aligned}$$