1 Introduction

Calculus III is a continuation of Calculus 2. The student is expected to be familiar with differentiation of the function of one real variable. In Calculus II, the inverse operator - antidifferentiation, also called integration, and many of its applications were covered.

In Calculus III - sequences, series, functions of several variables and multiple integration will be discussed. In addition, we provide, at an introductory level, other relevant and necessary topics such as vectors, matrices, determinants etc.

2 Infinite Sequences

2.1 Definition

Definition 2.1 If $f : N \to R$ be a function that assigns to each natural number $n \in N$ uniquely determined real number a_n, then the range of f written in the order

$$a_1,\ a_2,\ \ldots, a_n, \ldots \tag{2.1}$$

is called an *infinite sequence of real numbers* (or a *sequence of real numbers* or a *sequence in* R).

It is usual to use subscripts for the functional values of f instead of functional notations $f(1),\ f(2),\ \ldots$. We may indicate this assignment as:

$$\begin{array}{ccccccc} 1 & 2 & 3 & \ldots & n & \ldots \\ \downarrow & \downarrow & \downarrow & \ldots & \downarrow & \ldots \\ a_1 & a_2 & a_3 & \ldots & a_n & \ldots \end{array}$$

The real numbers so obtained are called the *elements* or the *terms* of the sequence, i.e. a_1 is the first element, a_2 is the second element of the sequence etc. The element a_n is called the *general element* or the *general term* or the nth *element* of the sequence.

A sequence may be represented in one of the following two ways:

1. By listing the elements

$$a_1,\ a_2,\ \ldots, a_n, \ldots$$

2. By giving the formula for the general term of the sequence, for example,

$$\{a_n\}_{n=1}^{\infty} = \{n^2+1\}_{n=1}^{\infty} = 2,\ 5,\ 10, \ldots$$

Remark It is important to note that the notation $\{a_n\}_{n=1}^{\infty}$ means more than just the set of points $\{a_n\ :\ n \in N\}$ which are all elements of the sequence; it denotes these points with the ordering that they get from the set of natural numbers. For example, the sequence $1,\ 3,\ 1,\ 3, \ldots$ consists of many 1's and 3's, all different in the sense that each corresponds to a different natural number. Rather, the set of all elements of the given sequence is equal to the set $\{1,\ 3\}$. Notice that the sequence $\{a_n\}_{n=1}^{\infty} = 1,\ 3,\ 1,\ 3, \ldots$ is not equal to the sequence $\{b_n\}_{n=1}^{\infty} = 3,\ 1,\ 3,\ 1, \ldots$ but the set of elements of both sequences is the set $\{1,\ 3\}$ ($\{1,\ 3\} = \{3,\ 1\}$).

Examples

(a) If the general element is $a_n = c$, where $c = const$, then the sequence $\{a_n\}_{n=1}^{\infty} = c,\ c,\ c,\ \ldots$ is a *constant sequence.*

(b) If $a_n = 3n + 2$, then

$$a_1 = 3.1 + 2 = 5, \qquad a_2 = 3.2 + 2 = 8, \qquad a_3 = 3.3 + 2 = 11$$

and thus, $\{a_n\}_{n=1}^{\infty} = 5,\ 8,\ 11,\ 14,\ 17, \ldots$

(c) If $a_n = \dfrac{1+(-1)^n}{2}$, then the sequence becomes $\{a_n\}_{n=1}^{\infty} = 0, 1, 0, 1, \ldots$

(d) The nth element of the sequence $1,\ \frac{1}{2},\ \frac{1}{3},\ \ldots,\ \frac{1}{n},\ \ldots$ is $a_n = \frac{1}{n}$.

(e) Since the elements of the sequence $2,\ 4,\ 6,\ \ldots,\ 2n,\ \ldots$ are even numbers then the nth element is $a_n = 2n$.

2.2 The Limit of a Sequence

Since a sequence is a function whose domain is the set of natural numbers, we can find limits of sequences as we learned in Calculus 1. For example, in order to find $\lim_{n\to\infty}\{\frac{1}{n}\}$, we note that we are evaluating $\lim_{n\to\infty} f(n)$, where $f(n) = \frac{1}{n}$. Thus, $\lim_{n\to\infty}\{\frac{1}{n}\} = 0$. Notice that this is similar to $\lim_{x\to\infty}\{\frac{1}{x}\}$, where x is a real number and not necessarily an integer. If $\lim\limits_{n\to\infty} a_n$ exists, we say that the sequence $\{a_n\}_{n=1}^{\infty}$ *converges* or is *convergent*. If $\lim\limits_{n\to\infty} a_n$ does not exist, we say that the sequence $\{a_n\}_{n=1}^{\infty}$ *diverges* or is *divergent*.

Using our knowledge of limits from Calculus 1, the following examples can be verified easily:

Example 1. If $\{a_n\}_{n=1}^{\infty}$ and $a_n = c$ (c is a constant) for all $n \in N$, then $\lim\limits_{n\to\infty} c = c$.

Example 2. Let $a_n = q^n$, $q \in R$.

(a) If $q = 0$, then $q^n = 0$ for all $n \in N$ and, by Example 1, $\lim\limits_{n\to\infty} q^n = 0$.

(b) Let $|q| < 1$, $q \neq 0$. Then $\lim\limits_{n\to\infty} q^n = 0$.

(c) If $q = 1$, then $q^n = 1$ for all $n \in N$ and, by Example 1, $\lim\limits_{n\to\infty} q^n = 1$.

(d) If $q = -1$ and $|q| > 1$, then the sequence $\{q^n\}_{n=1}^{\infty}$ is divergent.

From Example 2 it follows that:
The sequence $\{q^n\}_{n=1}^{\infty}$ is convergent if $-1 < q \leq 1$ and divergent for all other values of q and

$$\lim_{n\to\infty} q^n = \begin{cases} 0 & if \quad -1 < q < 1 \\ 1 & if \quad q = 1. \end{cases} \tag{2.2}$$

Example 3. Let $a_n = cos\left(\frac{1}{n}\right)$ and $f(x) = cos\left(\frac{1}{x}\right)$. Then we have that $\lim\limits_{x\to\infty} cos\left(\dfrac{1}{x}\right) = \cos 0 = 1$, hence, $\lim\limits_{n\to\infty} cos\left(\dfrac{1}{n}\right) = \cos 0 = 1$.

Example 4. We already know from Calculus 2 (Properties of the exponential function e^x) that $\lim\limits_{x\to\infty}\left(1 + \dfrac{1}{x}\right)^x = e$. If we put $n = x$, then an alternative expression for e is

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e. \tag{2.3}$$

Example 5. Calculate $\lim_{n\to\infty}\frac{1+\ln n}{\sqrt{n}}$. Notice that both numerator and denominator become infinite as $n\to\infty$. Thus $\lim_{n\to\infty}\frac{1+\ln n}{\sqrt{n}}$ is an indeterminate form of the type ∞/∞. We can apply l'Hospital's Rule to the related differentiable function $f(x)=\frac{1+\ln x}{\sqrt{x}}$ and obtain

$$\lim_{x\to\infty}\frac{1+\ln x}{\sqrt{x}} = \lim_{x\to\infty}\frac{1/x}{(\frac{1}{2})x^{-1/2}} = 2\lim_{x\to\infty}\frac{1}{x^{1/2}} = 0.$$

Therefore, $\lim_{n\to\infty}\frac{1+\ln n}{\sqrt{n}} = 0.$

Since sequences are functions whose domain is the set of natural numbers they satisfy the Limit Laws for the functions. In the next theorem, we repeat some properties of limits from Calculus 1 in this new context of a sequence.

Theorem 2.1

Assuming that the limits exist, then we have:

1. $\lim_{n\to\infty} ka_n = k\lim_{n\to\infty} a_n$ *for any constant* k.
2. $\lim_{n\to\infty}[a_n+b_n] = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.
3. $\lim_{n\to\infty}[a_n.b_n] = \lim_{n\to\infty} a_n . \lim_{n\to\infty} b_n$.
4. *If* $\lim_{n\to\infty} b_n \neq 0$, *then* $\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$.
5. $\lim_{n\to\infty}\sqrt[k]{a_n} = \sqrt[k]{\lim_{n\to\infty} a_n}$ *whenever the kth root is defined; in particular, if* $a_n \geq 0$ *for all* $n\in N$ *when* k *is even.*

Example 6.

$$\lim_{n\to\infty}\frac{2n^3}{7n^3+2n^2-1} = \frac{2}{7}.$$

To see why, we divide the numerator and denominator by n^3 and then take the limit. So,

$$\lim_{n\to\infty} \frac{2n^3}{7n^3+2n^2-1} = \lim_{n\to\infty} \frac{2n^3}{n^3(7+2\frac{1}{n}-\frac{1}{n^3})} = \lim_{n\to\infty} \frac{2}{7+2\frac{1}{n}-\frac{1}{n^3}} = \frac{2}{7}.$$

Example 7. Using Theorem 2.1 and the fact that $\lim\limits_{n\to\infty} q^n = 0$ for $|q| < 1$ we obtain

$$\lim_{n\to\infty} \frac{2^n+3^n}{4^n} = \lim_{n\to\infty} \frac{2^n}{4^n} + \lim_{n\to\infty} \frac{3^n}{4^n} = \lim_{n\to\infty} \left(\frac{1}{2}\right)^n + \lim_{n\to\infty} \left(\frac{3}{4}\right)^n = 0.$$

Example 8. Using the limit (2.3) we calculate

$$\lim_{n\to\infty} \left(\frac{n+2}{n}\right)^n = \lim_{n\to\infty} \left(\frac{n+2}{n+1}\right)^n \left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n+1}\right)^n \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \frac{\lim\limits_{n\to\infty} \left(1+\dfrac{1}{n+1}\right)^{n+1}}{\lim\limits_{n\to\infty} \left(1+\dfrac{1}{n+1}\right)}.e = e^2.$$

The Squeeze Theorem can be also used for the sequences as follows.

Theorem 2.2 *If $a_n \le c_n \le b_n$ for $n \ge M$, and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = a$, then $\lim\limits_{n\to\infty} c_n = a$.*

Example 9. $\lim\limits_{n\to\infty} \dfrac{sin^2 n}{3^n} = 0$. Since $0 \le sin^2 n \le 1$, then

$$0 \le \frac{sin^2 n}{3^n} \le \frac{1}{3^n}.$$

Also we have by (2.2) that $\lim\limits_{n\to\infty} \dfrac{1}{3^n} = 0$. Thus, $\lim\limits_{n\to\infty} \dfrac{sin^2 n}{3^n} = 0$ by the Squeeze Theorem.

Example 10. Find $\lim\limits_{n\to\infty} \dfrac{2^n}{n!}$. We will note that both numerator and denominator become infinite as $n \to \infty$. Thus $\lim\limits_{n\to\infty} \dfrac{2^n}{n!}$ is an indeterminate form of the type ∞/∞. But we can not apply l'Hospital's Rule, because here we have no corresponding differentiable function. ($n!$ is defined only for the non-negative integers by $n! = 1.2.3\ldots n$). We have that

$$0 < a_n = \frac{2^n}{n!} = \frac{2.\,2.\,2\ldots 2.\,2}{1.\,2.\,3\ldots(n-1).\,n} < \frac{2.\,2}{1.\,n} = \frac{4}{n}, \quad n = 4,\,5,\ldots$$

Since $\lim_{n\to\infty} \frac{4}{n} = 0$, then $\lim_{n\to\infty} \frac{2^n}{n!} = 0$ by the Squeeze Theorem.

Another useful fact for the limits of sequences is given by the following theorem, which can be verified easily from the inequality $-|a_n| \leq a_n \leq |a_n|$, and the Squeeze Theorem.

Theorem 2.3 *If* $\lim_{n\to\infty} |a_n| = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Example 11. $\lim_{n\to\infty} (-1)^n \frac{n^2}{2n^3+1} = 0.$

Since $|(-1)^n \frac{n^2}{2n^3+1}| = \frac{n^2}{2n^3+1}$, and $\lim_{n\to\infty} \frac{n^2}{2n^3+1} = 0$, then

$$\lim_{n\to\infty} (-1)^n \frac{n^2}{2n^3+1} = 0$$

by Theorem 2.3.

2.3 Bounded Sequences

Definition 2.2 A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be:
(a) *bounded above* if there exists a real number L such that $a_n \leq L$ *for all* $n \in N$;

(b) *bounded below* if there exists a real number l such that $a_n \geq l$ *for all* $n \in N$;

(c) *bounded* if it is bounded above and bounded below.

Examples

(a) The sequence $\{\frac{n}{n+1}\}_{n=1}^{\infty} = \frac{1}{2},\, \frac{2}{3},\, \frac{3}{4}, \ldots$ is bounded above by 1 since $\frac{n}{n+1} < 1 \iff n < n+1 \iff 0 < 1$, which is true. Also it is bounded below by $a_1 = \frac{1}{2}$. Therefore, the sequence $\{\frac{n}{n+1}\}_{n=1}^{\infty}$ is bounded.

(*b*) The sequence $\{n\}_{n=1}^{\infty}$ is bounded below by 1, but it is not bounded above.

The following theorem, which can be easily verified, is useful.

Theorem 2.4 *Every convergent sequence is bounded.*

Example 1. The sequence $\{2n-1\}_{n=1}^{\infty}$ is not bounded. Therefore, it is divergent. (If we suppose that the sequence is convergent, then by Theorem 2.4 it must be bounded.)

2.4. Monotonic Sequences

Definition 2.3 A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be:

(a) *nondecreasing* if $a_n \leq a_{n+1}$ *for all* $n \in N$. It is increasing if $a_n < a_{n+1}$.

(b) *nonincreasing* if $a_n \geq a_{n+1}$ *for all* $n \in N$. It is decreasing if $a_n > a_{n+1}$

(c) *monotonic* if it is either nonincreasing or nondecreasing.

A sequence is said to be *increasing* or *decreasing* if the inequalities in Definition 2.3 are strict.

Examples

(*a*) Prove that the sequence $\{\frac{n}{n+1}\}_{n=1}^{\infty}$ is increasing, that is, $a_n < a_{n+1}$ for all $n \in N$.

Solution 1. We have that $a_{n+1} = \frac{n+1}{n+2}$. Thus $a_n < a_{n+1} \iff \frac{n}{n+1} < \frac{n+1}{n+2}$ $\iff n(n+2) < (n+1)^2 \iff n^2+2n < n^2+2n+1 \iff 0 < 1$ which is true for all $n \in N$.

Solution 2. We can use the corresponding differentiable function $f(x) = \frac{x}{x+1}$ and evaluate

$$f'(x) = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2} > 0 \quad \textit{whenever} \quad x \neq -1.$$

Thus, $f(x)$ is increasing on $(-\infty,-1) \cup (-1,\infty)$ which implies that $f(n) < f(n+1)$ for all $n \in N$. Therefore, $\{\frac{n}{n+1}\}_{n=1}^{\infty}$ is increasing.

(*b*) The sequence $\{\frac{n!}{n^n}\}_{n=1}^{\infty}$ is decreasing. Indeed, $a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$ and $a_n > a_{n+1} \iff \frac{n!}{n^n} > \frac{(n+1)!}{(n+1)^{n+1}} \iff \frac{1}{n^n} > \frac{n+1}{(n+1)^{n+1}} \iff \frac{1}{n^n} > \frac{1}{(n+1)^n}$ $\iff \left(\frac{1}{n}\right)^n > \left(\frac{1}{n+1}\right)^n$, which is true for all $n \in N$ since $\frac{1}{n} > \frac{1}{n+1}$ for all $n \in N$.

We already know from Theorem 2.4 that every convergent sequence is bounded. But not every bounded sequence is convergent. For example, the sequence $\{a_n\}_{n=1}^{\infty}$, $a_n = (-1)^n$ is bounded because $-1 \leq a_n \leq 1$ for all $n \in N$ but it is divergent by (2.2). In the next theorem we will give conditions under which a bounded sequence is convergent.

Theorem 2.5 (Monotonic Sequence Criterion) *Every bounded and monotonic sequence is convergent.*

Theorem 2.5 shows that *every nondecreasing sequence, bounded above, is convergent.* (Likewise, *every nonincreasing sequence, bounded below, is convergent.*)

Example 1. We already know that the sequence $\{\frac{n}{n+1}\}_{n=1}^{\infty}$ is increasing and bounded above by 1. Therefore, it is convergent by Monotonic Sequence Criterion. The Monotonic Sequence Criterion doesn't tell us what the value of the limit is. But in this case,

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n}{n(\frac{1}{n}+1)} = \lim_{n\to\infty} \frac{1}{\frac{1}{n}+1} = 1.$$

2.5. Subsequences

Definition 2.4 A sequence $\{a_{n_k}\}_{k=1}^{\infty}$ which is formed from the elements of the sequence $\{a_n\}_{n=1}^{\infty}$ with indices n_1, n_2, ... such that $n_1 < n_2 < \ldots < n_k < n_{k+1} < \ldots$ is said to be a *subsequence* of $\{a_n\}_{n=1}^{\infty}$.

Notice that the sequence $\{a_n\}_{n=1}^{\infty}$ is a subsequence of itself. All other subsequences of $\{a_n\}_{n=1}^{\infty}$ are obtained by deleting certain elements from $\{a_n\}_{n=1}^{\infty}$ and leaving those remaining in their original relative order, i.e the sequence of their indices $\{n_k\}_{k=1}^{\infty}$ must be an increasing sequence of integers. A subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ is again a sequence.

Example 1. Consider the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$ with the nth element $a_n = \frac{1}{n}$.

(a) The following sequences

- $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots, \frac{1}{2k}, \ldots$ with the general element $b_k = a_{n_k} = \frac{1}{2k}$
- $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{k+1}, \ldots$ with the general element $b_k = a_{n_k} = \frac{1}{k+1}$
- $1, \frac{1}{4}, \frac{1}{9}, \ldots, \frac{1}{k^2}, \ldots$ with the general element $b_k = a_{n_k} = \frac{1}{k^2}$

are subsequences of the given sequence;

(b) The following sequences

- $1, \frac{1}{3}, 1, \frac{1}{3}, \ldots,$ with the general element

$$b_k = \begin{cases} 1 & if \quad k \; is \; odd \\ \frac{1}{3} & if \quad k \; is \; even \end{cases}$$

- $1, 2, 3, 2, 5, 6, 7, 2, 9, 10, \ldots,$ with the general element

$$b_k = \begin{cases} 2 & if \quad k = 2^m, \quad m \in N \\ k, & otherwise \end{cases}$$

are not subsequences of the given sequence

The following theorem is useful.

Theorem 2.6 *If a sequence $\{a_n\}_{n=1}^{\infty}$ converges to a real number a, then any subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ converges to a.*

Some times, the use of subsequences makes it easy to prove the divergence of a sequence.

Example 2. The sequence $\{a_n\}_{n=1}^{\infty} = \{\frac{1+(-1)^n}{2}\}_{n=1}^{\infty} = 0, 1, 0, 1, \ldots$ is divergent. If we suppose that the sequence is convergent to a number a, then by Theorem 2.6 every subsequence of $\{a_n\}_{n=1}^{\infty}$ must converge to a. Since there is a subsequence 0, 0, 0, ... converging to 0 and another subsequence 1, 1, 1, ... converging to 1, we conclude that our assumption is false. Therefore, $\{\frac{1+(-1)^n}{2}\}_{n=1}^{\infty}$ must be divergent.

Example 3. The sequence 1, 2, 1/3, 4, 1/5, 6, ..., with the general element

$$a_n = \begin{cases} \frac{1}{n} & if \quad n \, is \, odd \\ n & if \quad n \, is \, even \end{cases}$$

is divergent. Although a subsequence 1, 1/3, 1/5, ... of the given sequence converges to 0 there is another subsequence 2, 4, 6, ... which is divergent since it is unbounded. Therefore, 1, 2, 1/3, 4, 1/5, 6, ... is divergent.